STAT41500 **(Always take notes)**

Final Project

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1 Introductionn

When we learned the Lasso this kind of regression problem in high dimension setting, we need to assume the underlying true parameter is sparse in order to recover the parameter vector, otherwise we may have more parameter than sample sizes. Now we are aimed to recover the underlying matrix, but how to characterize the dimension of the parameter in a matrix? The answer is SVD, the number of parameter of a unknown matrix is equal to the number of parameter in its corresponding SVD, for a $n_1 \times n_2$ matrix, the number of parameter is $r(n_1 + n_2 - r)$, in low dimension is OK since our sample size N is enough to estimate these parameter, but in high dimension, if the underlying matrix is full rank, i.e. rank=min$(n_1, n_2) = r$, then the number of parameter $r(n_1 + n_2 - r) \gg N$, we cannot estimate the parameter, so the key assumption in matrix estimation is "low rank", which is analogous to sparsity $s \ll p$ in vector estimation.

I first teach the general low rank matrix estimation, then goes to the sample matrix completion, i.e Netflix recommendation problem. Finally, we learn a algorithm to recover the low rank matrix.

2 Low-rank matrix estimation

This section we will talk about the generalization of linear regression, which can be called matrix regression,i.e the unknown parameter from vector $\beta^* \in \mathbb{R}^p$ to matrix matrix $M^* \in \mathbb{R}^{p_1 x p_2}$.

Each observation has the form (A_i, y_i) =(covariate,response), A_i has same dimension with M^* and

$$y_i = \langle A_i, M^* \rangle + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2), i = 1, 2 \dots N$$

where $\langle A_i, M^* \rangle = Tr(A_i^T M^*)$, then we define observation operator $\mathcal{A} : \mathbb{R}^{p_1 x p_2} \to \mathbb{R}^N$ such as $\mathcal{A}(M^*)$ is a vector with N entries, i.e. $[\mathcal{A}(M^*)]_i = \langle A_i, M^* \rangle$

Write down our model:

$$y = \mathcal{A}(M^*) + \vec{\epsilon}, \quad \vec{\epsilon} \sim N(0, \sigma^2 I_N)$$

Where M^* is underlying true parameter matrix, with rank$(M^*) \leq r$.

Our estimator is given by:

$$\widehat{M} \in \arg \min_{M \in \mathbb{R}^{p_1 \times p_2}} \left\{ \frac{1}{2N} \|y - \mathcal{A}(M)\|_2^2 + \lambda \|M\|_* \right\}$$

The reason why we use nuclear norm to encourage sparsity is analogous to the ℓ_1 in lasso, as the computation for rank is NP hard, so we choose its convex relaxation, which is nuclear norm as our penalty.

Let's start our analysis from the basic inequality:

$$\frac{1}{2N} \|y - \mathcal{A}(\hat{M})\|_2^2 + \lambda \|\hat{M}\|_* \leq \frac{1}{2N} \|y - \mathcal{A}(M^*)\|_2^2 + \lambda \|M^*\|_*$$

$\Longrightarrow$

$$\begin{aligned}\frac{1}{2N}\|\mathcal{A}(\hat{M}-M^*)\|_2^2 &\le |\frac{1}{N}\langle\vec{\epsilon},\mathcal{A}(\hat{M}-M^*)\rangle| + \lambda(\|M^*\|_* - \|\hat{M}\|_*)\\ &= |\frac{1}{N}\langle\mathcal{A}^*(\vec{\epsilon}),\hat{M}-M^*\rangle| + \lambda(\|M^*\|_* - \|\hat{M}\|_*)\end{aligned}$$

Where $\mathcal{A}^*(\vec{\epsilon}) = \sum_{i=1}^{N}\epsilon_i A_i$

Lemma 1 $\langle X, Y\rangle \le \|X\|_*\|Y\|_{op}$
Proof:
Let SVD: $X = VDW$, V, W are unitary matrix.

$$\begin{aligned}\langle X, Y\rangle = Tr(Y^TX) = Tr(Y^TVDW) &= Tr(\left[(VYW)^TD\right])\\ = \sum_{k=1}^{p}\sum_{j=1}^{p}(VYW)^T_{kj}D_{jk} &= \sum_{k=1}^{p}(VYW)^T_{kk}\sigma_k(X)\\ \le \|VYW\|_{op}\sum_{k=1}^{p}\sigma_k(X) &= \|X\|_*\|Y\|_{op}\end{aligned}$$

Then define $\Delta = M^* - \hat{M}$ we can get

$$(1):\quad \frac{1}{2N}\|\mathcal{A}(\Delta)\|_2^2 \le \frac{1}{N}\|\mathcal{A}^*(\vec{\epsilon})\|_{op}\|\Delta\|_* + \lambda(\|M^*\|_* - \|\hat{M}\|_*)$$

Lemma 2 *Let rank$(M^*) \le r, \Delta = M^* - \widehat{M}$, then exist $\Delta_1, \Delta_2 \in \mathbb{R}^{p_1xp_2}$ such that*

1. $\Delta = \Delta_1 + \Delta_2$
2. *rank*$(\Delta_1) \le$ *2 rank*$(M^*) \le 2r$
3. *0*$=M^{*T}\Delta_2 = M^*\Delta_2^T = \langle\Delta_1, \Delta_2\rangle$

***Proof**: In Appendix Proof 1*

Lemma 3 *Let A and B be matrices of the same dimensions. If $AB' = 0$ and $A'B = 0$ then $\|A+B\|_* = \|A\|_* + \|B\|_*$.*
Proof: *SVD*

$$A = \begin{bmatrix} U_{A1} & U_{A2}\end{bmatrix}\begin{bmatrix}\Sigma_A & \\ & 0\end{bmatrix}\begin{bmatrix} V_{A1} & V_{A2}\end{bmatrix}^T \quad B = \begin{bmatrix} U_{B1} & U_{B2}\end{bmatrix}\begin{bmatrix}\Sigma_B & \\ & 0\end{bmatrix}\begin{bmatrix} V_{B1} & V_{B2}\end{bmatrix}^T$$

$AB^T = 0 \Longrightarrow V_{A_1}^TV_{B_1} = 0, A^TB = 0 \Longrightarrow U_{A_1}^TU_{B_1} = 0$. *Hence, there exist matrices U_C and V_C such that $[U_{A1}U_{B1}U_C]$ and $[V_{A1}V_{B1}V_C]$ are orthogonal. Then*

$$A = \begin{bmatrix} U_{A1} & U_{B1} & U_C\end{bmatrix}\begin{bmatrix}\Sigma_A & & \\ & 0 & \\ & & 0\end{bmatrix}\begin{bmatrix} V_{A1} & V_{B1} & V_C\end{bmatrix}^T$$

$$B = \begin{bmatrix} U_{A1} & U_{B1} & U_C\end{bmatrix}\begin{bmatrix}0 & & \\ & \Sigma_B & \\ & & 0\end{bmatrix}\begin{bmatrix} V_{A1} & V_{B1} & V_C\end{bmatrix}^T$$

Thus,

$$A + B = \begin{bmatrix} U_{A1} & U_{B1}\end{bmatrix}\begin{bmatrix}\Sigma_A & \\ & \Sigma_B\end{bmatrix}\begin{bmatrix} V_{A1} & V_{B1}\end{bmatrix}^T$$

Hence, $\|A+B\|_* = \|A\|_* + \|B\|_*$
Remark: If the row and column spaces of A and B are orthogonal, then $\|A+B\|_* = \|A\|_* + \|B\|_*$

Lemma 4 *Based on Lemma 2 and Lemma 3, and* $\Delta = M^* - \hat{M}$*, we have*

1. $\|\hat{M}\|_* - \|M^*\|_* \geq \|\Delta_2\|_* - \|\Delta_1\|_*$

$$\begin{aligned} \textit{since} \quad \|\hat{M}\|_* &= \|M^* - \Delta\|_* = \|M^* - \Delta_1 - \Delta_2\|_* \\ &\geq \|M^* - \Delta_2\|_* - \|\Delta_1\|_* = \|M^*\| + \|\Delta_2\|_* - \|\Delta_1\|_* \end{aligned}$$

2. *Triangle inequality:* $\|\Delta\|_* \leq \|\Delta_1\|_* + \|\Delta_2\|_*$
3. $\|\Delta_1\|_* \leq \sqrt{2r}\|\Delta_1\|_F \leq \sqrt{2r}\|\Delta\|_F$

Continued, we have

$$\begin{aligned} (1): \qquad \frac{1}{2N}\|\mathcal{A}(\Delta)\|_2^2 &\leq \frac{1}{N}\|\mathcal{A}^*(\vec{\epsilon})\|_{op}(\|\Delta_1\|_* + \|\Delta_2\|_*) + \lambda(\|\Delta_1\|_* - \|\Delta_2\|_*) \\ &= \left(\lambda + \frac{1}{N}\|\mathcal{A}^*(\vec{\epsilon})\|_{op}\right)\|\Delta_1\|_* + \left(\frac{1}{N}\|\mathcal{A}^*(\vec{\epsilon})\|_{op} - \lambda\right)\|\Delta_2\|_* \end{aligned}$$

Theorem 1 *(useful later)*

- *If* $\lambda \geq \frac{1}{N}\|\mathcal{A}^*(\vec{\epsilon})\|_{op}$*, then* $\frac{1}{2N}\|\mathcal{A}(\Delta)\|_2^2 \leq 2\sqrt{2r}\lambda\|\Delta\|_F$
- *If* $\lambda \geq \frac{2}{N}\|\mathcal{A}^*(\vec{\epsilon})\|_{op}$*, then* $\frac{1}{2N}\|\mathcal{A}(\Delta)\|_2^2 \leq \frac{3}{2}\lambda\|\Delta_1\|_* - \frac{1}{2}\lambda\|\Delta_2\|_*$
  Thus we get ***cone condition:*** $\|\Delta_2\|_* \leq 3\|\Delta_1\|_*$

Theorem 2 (kappa) *based on theorem 1*
Define

$$\kappa = \inf_{\bar{\Delta}\neq 0, \|\bar{\Delta}_2\|_* \leq 3\|\bar{\Delta}_1\|_*} \frac{\frac{1}{\sqrt{N}}\|\mathcal{A}(\bar{\Delta})\|_2}{\|\bar{\Delta}\|_F} > 0$$

Then $\kappa^2\|\Delta\|_F^2 \leq \frac{1}{N}\|\mathcal{A}(\Delta)\|_2^2 \implies \|\Delta\|_F \leq \frac{\sqrt{2r}\lambda}{\kappa^2}$*, thus*

$$\|\hat{M} - M^*\|_F^2 \leq \frac{2r\lambda^2}{\kappa^4}$$

Next we need to find the high probability bound for $\frac{1}{N}\|\mathcal{A}^*(\vec{\epsilon})\|_{op}$.

Lemma 5 (net-cover) *In previous class, we talk about* ϵ_0-*net-cover for* $\mathbb{R}^p$*; the results follows*
Define $S^{p-1} = \{v \in \mathbb{R}^p : \|v\| = 1\}$,
Let $\{u_1, ...u_{m_1}\} \subseteq S^{p_1-1}$ *be an* $\epsilon-$*net-cover for* $\mathbb{R}^{p_1}$
Let $\{v_1, ...v_{m_2}\} \subseteq S^{p_2-1}$ *be an* $\epsilon-$*net-cover for* $\mathbb{R}^{p_2}$
We have shown in the class that we can take $\epsilon_0 = \frac{1}{4}, m_1 \leq 9^{p_1}, m_2 \leq 9^{p_2}$

Define $\frac{1}{N}\|\mathcal{A}^*(\vec{\epsilon})\|_{op} = \|\frac{1}{N}\sum_{k=1}^N \epsilon_k A_k\|_{op} := \|Q\|_{op}$, and we know $\|Q\|_{op} = \sup_{v\in S^{p_2-1}}\|Qv\|_2$.
For any $v \in S^{p_2-1}$, according to the definition of net, $\|Qv\|_2 = \|Q(v - v_j + v_j)\|_2 \leq \|Qv_j\|_2 + \|Q(v-v_j)\|_2 \leq \max_{1\leq j\leq m_2}\|Qv_j\|_2 + \epsilon_0\|Q\|_{op}$
For any $u \in S^{p_1-1}$,$\langle u, Qv_j\rangle = \langle u - u_i + u_i, Qv_j\rangle = \langle u - u_i, Qv_j\rangle + \langle u_i, Qv_j\rangle \leq \epsilon_0\|Qv_j\|_2 + \max_{1\leq i\leq m_1}\langle u_i, Qv_j\rangle \leq \epsilon_0\|Q\|_{op} + \max_{1\leq i\leq m_1}\langle u_i, Qv_j\rangle$
Take $\epsilon_0 = \frac{1}{4}$, we have

$$\|Q\|_{op} \leq \frac{1}{2}\|Q\|_{op} + \max_{1\leq j\leq m_2}\max_{1\leq i\leq m_1}\langle u_i, Qv_j\rangle$$

Which imply

$$\|Q\|_{op} \leq 2\max_{1\leq j\leq m_2}\max_{1\leq i\leq m_1}\langle u_i, Qv_j\rangle$$

Let us assume $\max_k |\langle u_i, A_k v_j\rangle| \leq 1$(or O(1)), then we have

$$\begin{aligned}\mathbb{P}\left[\frac{1}{N}\|\mathcal{A}^*(\epsilon)\|_{op}\right] &\leq \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\mathbb{P}\left(|\langle u_i, Qv_j\rangle| > \frac{t}{2}\right)\\ &\leq \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\mathbb{P}\left(|\frac{1}{N}\sum_{k=1}^{N}\epsilon_k\langle u_i, A_k v_j\rangle| > \frac{t}{2}\right)\\ &\leq \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\mathbb{P}\left(|N(0,\frac{\sigma^2}{N})| > \frac{t}{2}\right)\\ &\leq 2\exp\left(-C\frac{Nt^2}{\sigma^2} + (p_1+p_2)\log 9\right)\end{aligned}$$

Take t=$C\sigma\sqrt{\frac{p_1+p_2}{N}}$, we have

$$\frac{1}{N}\|\mathcal{A}^*(\epsilon)\|_{op} \lesssim \sigma\sqrt{\frac{p_1+p_2}{N}} \quad w.h.p$$

Theorem 3 *Assume* $\max_k |\langle u_i, A_k v_j\rangle| \leq 1$*(or O(1)), and set* $\lambda = C\sigma\sqrt{\frac{p_1+p_2}{N}}$*, according to Theorem 2 we have*

$$\left\|\hat{M} - M^*\right\|_F^2 \lesssim \frac{r\sigma^2(p_1+p_2)}{N\kappa^4} \quad w.h.p$$

But the value of κ may be very small, we can assume observation operator $\mathcal{A}$ is from Gaussian distribution in order to remove κ, which is analogous to the random design in lasso problem.

Theorem 4 *Suppose the entries of* A_k *is from iid N(0,1) and assume* $\frac{r(p_1+p_2)}{N}$ *is sufficient small, then we have*

① $\max_k |\langle u_i, A_k v_j\rangle| = O(1)$ *w.h.p*

② *Under cone condition:* $\kappa \gtrsim 1$ *w.h.p*

***Proof of* ①:** *Since* $\langle u_i, A_k v_j\rangle \sim N(0,1)$*, then* $|\langle u_i, A_k v_j\rangle| \leq 10$ *with high probability.*

***Proof of* ②:**

Recall the definition of κ *is :* $\kappa = \inf_{\|X\|_F\neq 0}\frac{\frac{1}{\sqrt{N}}\|\mathcal{A}(X)\|_2}{\|X\|_F} = \inf_{\|X\|_F=1}\frac{1}{\sqrt{N}}\|\mathcal{A}(X)\|_2$*, rewrite* $\|\mathcal{A}(X)\|_2 = \sup_{u\in S^{N-1}}\langle u, \mathcal{A}(X)\rangle$*. Then* $\kappa = \inf_{\|X\|_F=1}\sup_{u\in S^{N-1}}\langle u, \mathcal{A}(X)\rangle$*.*

Define $Z_{u,X} = \langle u, \mathcal{A}(X)\rangle$ *is Gaussian R.V with zero mean. For any two pairs* $(u, X), (u', X')$

$$\begin{aligned}(Z_{u,X} - Z_{u',X'})^2 &= \left[\left\langle\begin{pmatrix}u_1\\u_2\\\vdots\\u_N\end{pmatrix}, \begin{pmatrix}\langle A_1, X\rangle\\\langle A_2, X\rangle\\\vdots\\\langle A_N, X\rangle\end{pmatrix}\right\rangle - \left\langle\begin{pmatrix}u_1'\\u_2'\\\vdots\\u_N'\end{pmatrix}, \begin{pmatrix}\langle A_1, X'\rangle\\\langle A_2, X'\rangle\\\vdots\\\langle A_N, X'\rangle\end{pmatrix}\right\rangle\right]^2\\ &= \sum_{i=1}^{N}(u_i\langle A_i, X\rangle - u_i'\langle A_i, X'\rangle)^2\end{aligned}$$

$\Longrightarrow \mathbb{E}(Z_{u,X} - Z_{u',X'})^2 = \|u\otimes X - u'X'\|_F^2$

Next define $Y_{u,X} = \langle w, u\rangle + \langle G, X\rangle$ *where entry of* w *and* G *are iid N(0,1), so* $Y_{u,X}$ *is Gaussian with zero mean.*

$$(Y_{u,X} - Y_{u',X'})^2 = \left[\left\langle\begin{pmatrix}w_1\\w_2\\\vdots\\w_N\end{pmatrix}, \begin{pmatrix}u_1-u_1'\\u_2-u_2'\\\vdots\\u_N-u_N'\end{pmatrix}\right\rangle + Tr[G^T(X-X')]\right]^2$$

$\Longrightarrow \mathbb{E}(Y_{u,X} - Y_{u',X'})^2 = \|u-u'\|_2^2 + \|X-X'\|_F^2$
And we have $\|u \otimes X - u'X'\|_F^2 \leq \|u-u'\|_2^2 + \|X-X'\|_F^2$ equality hold with $X = X'$

Lemma 6 (Gordon-Slepian inequality) *Let $X_{i,j}$ and $Y_{i,j}$($1 \leq i \leq n, 1 \leq j \leq m$) be real-valued centered Gaussian R.Vs which satisfy follwoing conditions:*

1. $\mathbb{E}(|Y_{i,j} - Y_{i,k}|^2) \leq \mathbb{E}(|X_{i,j} - X_{i,k}|)^2$ *for all* $1 \leq i \leq n, 1 \leq j,k \leq m$
2. $\mathbb{E}(|Y_{i,j} - Y_{l,k}|^2) \geq \mathbb{E}(|X_{i,j} - X_{l,k}|)^2$ *for all* $i \neq l$

Then

$$\mathbb{E}(\min_{1\leq i\leq n}\max_{1\leq j\leq m} Y_{i,j}) \leq \mathbb{E}(\min_{1\leq i\leq n}\max_{1\leq j\leq m} X_{i,j})$$

Apply the Lemma 6: Since we have shown $\mathbb{E}\left(Z_{u,X} - Z_{u',X'}\right)^2 \leq \mathbb{E}\left(Y_{u,X} - Y_{u',X'}\right)^2$ the condition 2 holds, condition 1 holds since we have equality when $X = X'$, so we can get

$$\mathbb{E}(\inf_{\|X\|_F=1}\sup_{u\in S^{N-1}} Z_{u,X}) \geq \mathbb{E}(\inf_{\|X\|_F=1}\sup_{u\in S^{N-1}} Y_{u,X})$$

Hence we get following relation:

$$\mathbb{E}[\inf_{\|X\|_F=1}\|\mathcal{A}(X)\|_2] = \mathbb{E}(\inf_{\|X\|_F=1}\sup_{u\in S^{N-1}} Z_{u,X}) \geq \mathbb{E}(\inf_{\|X\|_F=1}\sup_{u\in S^{N-1}} Y_{u,X})$$

Lemma 7 *Let $Z_1, Z_2, \ldots Z_n \overset{iid}{\sim} N(0,1)$, then $Y = \sqrt{Z_1^2 + Z_2^2 + \ldots Z_n^2} \sim \chi_n$ with*

$$\mathbb{E}(Y) = \sqrt{2}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \geq \frac{1}{2}\sqrt{n} \quad \textit{(by MI)}$$

Continue our analysis, we compute

$$\begin{aligned}\mathbb{E}(\inf_{\|X\|_F=1}\sup_{u\in S^{N-1}} Y_{u,X}) &= \mathbb{E}[\sup_{u\in S^{N-1}}\langle w,u\rangle] + \mathbb{E}[\inf_{\|X\|_F=1}\langle G,X\rangle] \\ &= \mathbb{E}[\|w\|_2] - \mathbb{E}[\sup_{\|X\|_F=1}\langle G,X\rangle] \\ &\geq \frac{1}{2}\sqrt{N} - \mathbb{E}\|G\|_{op}\|X\|_*\end{aligned}$$

Proposition 1 *Suppose condition of Theorem 4 hold, then*

$$\inf_{\|X\|_F\neq 0}\frac{\frac{1}{\sqrt{N}}\|\mathcal{A}(X)\|_2}{\|X\|_F} \geq \frac{1}{2} - \frac{1}{\sqrt{N}}\mathbb{E}\|G\|_{op}\frac{\|X\|_*}{\|X\|_F}$$

By standard covering lemma, we have $\mathbb{E}\|G\|_{op} \leq C\sqrt{p_1+p_2}$ w.h.p. And inequality $\|X_*\| \leq \sqrt{rank(X)}\|X\|_F$, replace X by $\Delta = \hat{M} - M^*$, which satisfy $\|\Delta_2\|_* \leq 3\|\Delta_1\|_*, rank(\Delta_1) \leq 2r$, we can get

$$\|\Delta\|_* \leq \|\Delta_1\|_* + \|\Delta_2\|_* \leq 4\|\Delta_1\| \leq 4\sqrt{2r}\|\Delta_1\|_F \leq 4\sqrt{2r}\|\Delta\|_F \Longrightarrow \frac{\|\Delta\|_*}{\|\Delta\|_F} \leq 4\sqrt{2r}$$

$\Longrightarrow$

$$\kappa = \inf_{\Delta\neq 0,\|\Delta_2\|_*\leq 3\|\Delta_1\|_*}\frac{\frac{1}{\sqrt{N}}\|\mathcal{A}(\Delta)\|_2}{\|\Delta}\|_F \geq \frac{1}{2} - 4\sqrt{2}\sqrt{\frac{r(p_1+p_2)}{N}} \quad w.h.p$$

Since we assume that $\frac{r(p_1+p_2)}{N}$ is sufficiently small, we have $\kappa \gtrsim 1$ w.h.p.

Combine Theorem 3 and Theorem 4, we can get our final Theorem.

> **Theorem 5** *Assume all elements of observation operator $\mathcal{A}$ is from iid N(0,1), and assume $\frac{r(p_1+p_2)}{N}$ is sufficiently small, and rank(M^*) $\leq r$ then*
>
> $$\|\hat{M} - M^*\|_F^2 \lesssim \frac{\sigma^2 r(p_1+p_2)}{N} \quad w.h.p$$

3 Matrix completion

In matrix completion problems, the observation has the form $(\sqrt{p_1p_2}e_{a(i)}e_{b(i)}^T, y_i)$, which means that A_i has only one non-zero entry with value $\sqrt{p_1p_2}$. This is parallel to the Gaussian sequence model with $X = \sqrt{p}I$ so the matrix completion is a special case of low rank matrix estimation. Our model can be written as:

$$\widetilde{y}_i = M^*_{a(i),b(i)} + \frac{\epsilon_i}{\sqrt{p_1p_2}}, \quad \epsilon_i \sim N(0,\sigma^2), i = 1,2,\dots n$$

Where $\widetilde{y}_i$ is observed noisy entry of true matrix.

Or we may think each entry of M^* is picked with probability $\frac{1}{p_1p_2}$("observing" at random), then we have following relationship:

$$\mathbb{E}\left[\frac{\|\mathcal{A}(M^*)\|_2^2}{n}\right] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\langle A_i, M^*\rangle^2\right] = \frac{1}{n}\sum_{i=1}^n \|M^*\|_{\mathrm{F}}^2 = \|M^*\|_{\mathrm{F}}^2$$

But in high dimension setting,i.e. $n \ll p_1p_2$, in order to successfully recover underlying M^*, we need to have some conditions on M^* and observations. Think about the following bad cases:

- Case 1:

$$M^* := e_1 \otimes e_{p_2} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

  If we observe entry with probability $\frac{1}{p_1p_2}$, then it is highly possible that we only observe zero.

- Case 2: Suppose in the true matrix M^*, there are some values in each row and column, but in the observation, some row or column is unobserved. E.g: no information of second row,

$$\begin{bmatrix} a & b & c & d & e \\ ? & ? & ? & ? & ? \\ j & i & h & g & f \\ k & l & m & n & o \\ t & s & r & q & p \end{bmatrix}$$

  then it is impossible to recover M^*, intuitively if we have no data about a specific user, how can we guess/infer his preference?

Case 2 problem is not our concern if we treat the sampling as uniform and number of sample is sufficient, that is to say we always observe some values in each row and column.

For case 1 problem, literature pose incoherent assumption on unknown matrix M^*, which assume singular vectors of M^* are relatively "spread out"(if the singular vector is canonical basis e_i, then case 1 problem shown). There are various ways to address the deficiency of the coherence condition, in our analysis, we assume maximum absolute value of the matrix is bounded, i.e. $\|M^*\|_{max} \leq \frac{\alpha}{\sqrt{p_1p_2}}$. Our estimator is given by:

$$\widehat{M} \in \arg \min_{\|M\|_{\max} \leq \frac{\alpha}{\sqrt{d_1d_2}}} \left\{\frac{1}{2n}\|y - \mathcal{A}(M)\|_2^2 + \lambda\|M\|_*\right\}$$

Where y_i is ith observed noisy entry $\times\sqrt{p_1p_2}$

Following the same basic inequality, we get the Theorem 1 and Theorem 2. Next we need to find the high probability bound for $\frac{1}{n}\|\mathcal{A}^*(\vec{\epsilon})\|_{op}$

Since $\frac{1}{n}\|\mathcal{A}^*(\vec{\epsilon})\|_{op} = \|\frac{1}{n}\mathcal{A}^*(\vec{\epsilon})\|_{op} = \|\frac{1}{n}\sum_{i=1}^n \epsilon_i A_i\|_{op}$ with $\epsilon_i \sim N(0,\sigma^2)$. And notice $\mathbb{E}(\epsilon_i A_i) = \mathbb{E}(\epsilon_i)\mathbb{E}(A_i) = 0$. If we can show $\epsilon_i A_i$ is zero mean symmetric sub-gaussian, then we can apply Hoeffding bound for random matrix.

We check the moment generating function:

$$\mathbb{E}(e^{\lambda\epsilon_i A_i}) = \sum_{k=0}^{\infty}\frac{\lambda^k\mathbb{E}(\epsilon_i A_i)^k}{k!} = \sum_{k=0}^{\infty}\frac{\lambda^{2k}\mathbb{E}(\epsilon_i^{2k})\mathbb{E}(A_i^{2k})}{(2k)!} \overset{Jensen}{\leq} \sum_{k=0}^{\infty}\frac{\lambda^{2k}(\mathbb{E}\epsilon_i^2)^k(\mathbb{E}A_i^2)^k}{(2k)!} = e^{\frac{\lambda^2\sigma^2\mathbb{E}(A_i^2)}{2}}$$

But notice that $A_i \neq A_i^T$, so we construct $Q_i = \begin{bmatrix} \mathbf{O} & A_i \\ A_i^T & \mathbf{O} \end{bmatrix} \in \mathbb{R}^{(p_1+p_2)x(p_1+p_2)}$, and we have $\|\frac{1}{n}\sum_{i=1}^n \epsilon_i A_i\|_{op} = \|\frac{1}{n}\sum_{i=1}^n \epsilon_i Q_i\|_{op}$.

$$\mathbb{E}(e^{\lambda\epsilon_i Q_i}) \leq e^{\frac{\lambda^2\sigma^2\mathbb{E}(Q_i^2)}{2}}, \mathbb{E}(Q_i^2) = \frac{1}{p_1p_2}p_1p_2 diag(p_1, p_1 .... p_2, p_2 ... p_2) \preceq \max(p_1,p_2)I_{p_1+p_2}$$

So $\epsilon_i Q_i$ is sub-gaussian with $V_i = \sigma_2\max(p_1,p_2)I_{p_1+p_2}$

Lemma 8 (Hoeffding bound for random matrix) *Let $\{B_i\}_{i=1}^n$ be a sequence of zero-mean independent symmetric random matrices that satisfy the sub-gaussian condition with parameter $\{V_i\}_{i=1}^n$. Then for $\delta > 0$*

$$\mathbb{P}\left(\|\frac{1}{n}\sum_{i=1}^n B_i\|_{op} \geq \delta\right) \leq 2dim(B_i)e^{-\frac{n\delta^2}{2v^2}}$$

where $v^2 = \|\frac{1}{n}\sum_{i=1}^n V_i\|_{op}$

Apply Lemma 8 with $dim(\epsilon_i Q_i) = p_1 + p_2, v^2 = \|\frac{1}{n}\sum_{i=1}^n V_i\|_{op} = \|\sigma^2\max(p_1,p_2)I_{p_1+p_2}\|_{op} = \sigma^2\max(p_1,p_2)$ $\Longrightarrow$

$$\begin{aligned}\mathbb{P}\left(\|\frac{1}{n}\sum_{i=1}^n \epsilon_i A_i\|_{op}\right) &= \mathbb{P}\left(\|\frac{1}{n}\sum_{i=1}^n \epsilon_i Q_i\|_{op}\right)\\ &\leq 2(p_1+p_2)\exp(-\frac{nt^2}{2\max(p_1,p_2)\sigma^2})\\ &\leq 2\exp-\frac{Cnt^2}{(p_1+p_2)\sigma^2} + \log(p_1+p_2)\end{aligned}$$

Take $t^2 = C\frac{\sigma^2(p_1+p_2)log(p_1+p_2)}{n}$, we get

$$\frac{1}{n}\|\mathcal{A}^*(\vec{\epsilon})\|_{op}^2 \lesssim \frac{\sigma^2(p_1+p_2)log(p_1+p_2)}{n} \quad w.h.p$$

Theorem 6 *Assume $rank(M^*) \leq r$, set $\lambda = C\sigma\sqrt{\frac{(p_1+p_2)\log(p_1+p_2)}{n}}$, we have*

$$\left\|\hat{M} - M^*\right\|_F^2 \lesssim \frac{r\sigma^2(p_1+p_2)\log(p_1+p_2)}{n\kappa^4} \quad w.h.p$$

Next task is to lower bound the kappa: i.e lower bound $\frac{\frac{1}{n}\|\mathcal{A}(\Delta)\|_2^2}{\|\Delta\|_F^2}$, without loss of generality, define the set

$$S(\alpha,\rho) = \left\{\mathbf{\Theta} \in \mathbb{R}^{p_1\times p_2} | \|\mathbf{\Theta}\|_{\mathrm{F}} = 1, \quad \|\mathbf{\Theta}\|_{\max} \leq \frac{\alpha}{\sqrt{p_1p_2}} \quad \text{and} \quad \|\mathbf{\Theta}\|_* \leq \rho\right\}$$

Remark: under cone condition, $\rho = 4\sqrt{2r}$

Let $Z(\alpha,\rho) := \sup_{\mathbf{\Theta}\in\mathbb{S}(\alpha,\rho)}\left|\frac{1}{n}\|\mathcal{A}(\mathbf{\Theta})\|_2^2 - 1\right|$, we sim to show $Z(\alpha,\rho)$ is closed to zero.

Define $F_{\mathbf{\Theta}}(A) := \langle\Theta, A\rangle^2, |F_\Theta(A)| \leq \|\Theta\|_{\max}^2\|A\|_1^2 \leq \frac{\alpha^2}{p_1p_2}p_1p_2 = \alpha^2$

Lemma 9 (Functional Hoeffding theorem) *Let $Z = \sup_{f\in\mathcal{F}}\{\frac{1}{n}\sum^n f(X_i)\}$, and f is bounded, i.e. $f \in [-\frac{L}{2}, \frac{L}{2}]$, then*

$$\mathbb{P}(Z \geq \mathbb{E}(Z) + \delta) \leq \exp(-\frac{n\delta^2}{4L^2})$$

Take $\delta = C\alpha\sqrt{\frac{(p_1+p_2)\log(p_1+p_2)}{n}}$, we have

$$\mathbb{P}\left(Z(\alpha,\rho) \geq \mathbb{E}(Z(\alpha,\rho)) + C\alpha\sqrt{\frac{(p_1+p_2)\log(p_1+p_2)}{n}}\right) \leq \exp[-C(p_1+p_2)+\log(p_1+p_2)]$$

Lemma 10 (Rademacher symmetrization)

$$\mathbb{E}\left[\sup_{g\in\mathcal{G}}\left(\frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \mathbb{E}g(Z)\right)\right] \leq \mathbb{E}\left[\sup_{g\in\mathcal{G}}\frac{1}{n}\sum_{i=1}^{n}\epsilon_i' g(Z_i)\right]$$

where $\epsilon_i' = \begin{cases} 1 & \textit{with probability } 0.5 \\ -1 & \textit{with probability } 0.5 \end{cases}$

Then we have

$$\mathbb{E}(Z(\alpha,\rho)) \leq 2\mathbb{E}\left[\sup_{\boldsymbol{\Theta}\in\mathbb{S}(\alpha,\rho)} |\frac{1}{n}\sum_{i=1}^{n}\epsilon_i'\langle A_i,\boldsymbol{\Theta}\rangle^2|\right] \leq 2\alpha\mathbb{E}\left[\sup_{\boldsymbol{\Theta}\in\mathbb{S}(\alpha,\rho)} |\frac{1}{n}\sum_{i=1}^{n}\epsilon_i'\langle A_i,\boldsymbol{\Theta}\rangle|\right]$$

Since $|\langle A_i,\boldsymbol{\Theta}\rangle| \leq \alpha$ for all $\boldsymbol{\Theta}$ and A_i. And $\mathbb{E}\left[\sup_{\boldsymbol{\Theta}\in\mathbb{S}(\alpha,\rho)} |\frac{1}{n}\sum_{i=1}^{n}\epsilon_i'\langle A_i,\boldsymbol{\Theta}\rangle|\right] \leq \rho\mathbb{E}[\|\frac{1}{n}\sum_{i=1}^{n}\epsilon_i' A_i\|_{op}]$

The analysis of upper bound for $\mathbb{E}[\|\frac{1}{n}\sum_{i=1}^{n}\epsilon_i' A_i\|_{op}]$ is similar with what we have done after Lemma 8, i.e. $\mathbb{E}[\|\frac{1}{n}\sum_{i=1}^{n}\epsilon_i' A_i\|_{op}] \leq C\sqrt{\frac{(p_1+p_2)\log(p_1+p_2)}{n}}$ Thus we have

$$\mathbb{E}[Z(\alpha,\rho)] \leq C_1\alpha\rho\sqrt{\frac{(p_1+p_2)\log(p_1+p_2)}{n}}$$

We have derive high probability bound for $Z(\alpha,\rho)$:

$$Z(\alpha,\rho) := \sup_{\boldsymbol{\Theta}\in\mathbb{S}(\alpha,\rho)}\left|\frac{1}{n}\|\mathcal{A}(\boldsymbol{\Theta})\|_2^2 - 1\right| \leq C_1\alpha\rho\sqrt{\frac{(p_1+p_2)\log(p_1+p_2)}{n}} + C_2\alpha\sqrt{\frac{(p_1+p_2)\log(p_1+p_2)}{n}} \quad w.h.p$$

Replace $\rho = 4\sqrt{2r}$ and extend to general form, we can get

$$\begin{aligned}\left|\frac{\frac{1}{n}\|\mathcal{A}(\Delta)\|_2^2}{\|\Delta\|_F^2} - 1\right| &\leq C_1\alpha\|\Delta\|_F\sqrt{\frac{r(p_1+p_2)\log(p_1+p_2)}{n}} + C_2\alpha\|\Delta\|_F\sqrt{\frac{(p_1+p_2)\log(p_1+p_2)}{n}} \\ &\leq C_3\alpha\|\Delta\|_F\sqrt{\frac{r(p_1+p_2)\log(p_1+p_2)}{n}} \quad w.h.p\end{aligned}$$

for all $0 \neq \Delta \in \mathbb{R}^{p_1 x p_2}$ that satisfy the cone condition.

Based on above inequality, we have

$$\begin{aligned}\frac{1}{n}\|\mathcal{A}(\Delta)\|_2^2 &\geq \|\Delta\|_F^2 - C_3\alpha\|\Delta\|_F\sqrt{\frac{r(p_1+p_2)\log(p_1+p_2)}{n}} \\ &= \|\Delta\|_F\left(\|\Delta\|_F - C_3\alpha\sqrt{\frac{r(p_1+p_2)\log(p_1+p_2)}{n}}\right)\end{aligned}$$

- **Case 1**: if $\|\Delta\|_F \geq 2C_3\alpha\sqrt{\frac{r(p_1+p_2)\log(p_1+p_2)}{n}}$ then $\frac{1}{n}\|\mathcal{A}(\Delta)\|_2^2 \geq \frac{1}{2}\|\Delta\|_F^2 \Longrightarrow \kappa^2 \geq \frac{1}{2}$
  $\Longrightarrow \|\Delta\|_F^2 \lesssim \frac{r\sigma^2(p_1+p_2)\log(p_1+p_2)}{n}$ *w.h.p* By Theorem 6
- **Case 2**: if $\|\Delta\|_F \leq 2C_3\alpha\sqrt{\frac{r(p_1+p_2)\log(p_1+p_2)}{n}}$ then $\|\Delta\|_F^2 \lesssim \frac{r\alpha^2(p_1+p_2)\log(p_1+p_2)}{n}$ *w.h.p*

Theorem 7 *For model* $\tilde{y}_i = M^*_{a(i),b(i)} + \frac{\epsilon_i}{\sqrt{p_1p_2}}$, $\epsilon_i \sim N(0,\sigma^2)$. *Assume* $\|M^*\| \leq \frac{\alpha}{\sqrt{p_1p_2}}$ *and* $rank(M^*) \leq r$. *then*

$$\left\|\hat{M} - M^*\right\|_F^2 \lesssim \max(\alpha^2,\sigma^2)\frac{r(p_1+p_2)\log(p_1+p_2)}{n} \quad w.h.p$$

4 Computation for matrix completion

Suppose underlying true parameter M^* is of good structure and our observation is sufficient to estimate M^*. Then our estimator:

$$\hat{M} = arg \min_{B \in \mathbb{R}^{p_1 x p_2}} \frac{1}{2} \sum_{(i,j)\in\Omega} (Y_{ij} - B_{ij})^2 + \lambda \|B\|_*$$

Where Ω is the index of observed entries.

Define P_Ω, projection operator onto observed set:

$$[P_\Omega(B)]_{ij} = \begin{cases} B_{ij} & (i,j) \in \Omega \\ 0 & (i,j) \notin \Omega \end{cases}$$

$f(B) = \frac{1}{2} \|P_\Omega(Y) - P_\Omega(B)\|_F^2 + \lambda\|B\|_* = g(B) + h(B)$

Two items needed for proximal gradient descent:

- Gradient of g : $\nabla g(B) = -\left(P_\Omega(Y) - P_\Omega(B)\right)$
- Prox function: $\text{prox}_t(B) = \underset{Z}{\text{argmin}} \frac{1}{2t}\|B - Z\|_F^2 + \lambda\|Z\|_*$

Claim: $\text{prox}_t(B) = S_{\lambda t}(B)$, matrix soft-thresholding.

$S_\lambda(B) = U\Sigma_\lambda V^T$ where $B = U\Sigma V^T$ and $(\Sigma_\lambda)_{ii} = \max\{\Sigma_{ii} - \lambda, 0\}$

Proof:
if $Z = U\Sigma V^T$, then $\partial\|Z\|_* = \left\{UV^T + W : \|W\|_{\text{op}} \le 1, U^T W = 0, WV = 0\right\}$, ,note that $\text{prox}_t(B) = Z$, where Z need to satisfy

$$0 \in Z - B + \lambda t \cdot \partial\|Z\|_*$$

plug in $Z = U\Sigma_{\lambda t}V^T, r = rank(Z)$

$$\begin{aligned} Z - B + \lambda t \partial\|Z\|_* &= U\Sigma_{\lambda t}V^T - U\Sigma V^T + \lambda t(U_r V_r^T + W) \\ &= U(\Sigma_{\lambda t} - \Sigma + \lambda t)V^T + \lambda t W \\ &= 0 \quad \text{set } W = -U_{[r+1:n]}V_{[r+1:n]}^T \end{aligned}$$

So $B^{(t+1)} = S_{\lambda t}\left(B^{(t)} + t\left(P_\Omega(Y) - P_\Omega(B^{(t)})\right)\right)$

Note that $\nabla g(B)$ is Lipschitz continuous with $L = 1$, so we can choose fixed step size $t = 1$. Update step is now:

$$B^{(t+1)} = S_\lambda\left(P_\Omega(Y) + P_\Omega^\perp(B^{(t)})\right)$$

5 Appendix

Proof 1 *We write the SVD as* $M^* = UDV^T$, *U, V are unitary matrix and* $D = diag(\sigma_1, \sigma_2 \ldots, \sigma_r, 0, 0, ...0)$ $\Gamma = U^T \Delta V \in \mathbb{R}^{p_1 \times p_2}$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \quad \textit{where } \Gamma_{11} \in \mathbb{R}^{r \times r}, \ \textit{and } \Gamma_{22} \in \mathbb{R}^{(p_1 - r) \times (p_2 - r)}$$

Let

$$\Delta_2 = U \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_{22} \end{bmatrix} V^T, \textit{and } \Delta_1 = \Delta - \Delta_2$$

$$\operatorname{rank}(\Delta_1) = \operatorname{rank} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & 0 \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & 0 \end{bmatrix} + \operatorname{rank} \begin{bmatrix} 0 & 0 \\ \Gamma_{21} & 0 \end{bmatrix} \leq 2r$$

And it is obvious that $0 = M^{*T} \Delta_2 = M^* \Delta_2^T = \langle \Delta_1, \Delta_2 \rangle$.